1D hyperbolic model for coastal waves

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Introduction

In the context of climate change, the risks of flooding and coastal erosion are increased by numbers of extreme weather events. It is therefore essential to develop reliable, accurate and operational models to describe the phenomena of water wave propagating near the coast. For coastal erosion, wave dispersion and wave breaking are crucial to the evolution of bathymetry. As discussed in the paper of Kazakova and Richard 2019, how to precisely capture them is an issue of the model. Also, due to the non-hydrostatic pressure, some non-linear terms similar to those in the Serre-Green-Naghdi equations appear. Those terms contain mixed space-time third derivative which necessitate a time-consuming elliptic step in the numerical resolution. To avoid dealing with those terms directly, we replace them by new depth-averaged quantities. Such technique proposed in the paper of Richard 2021 is applied here to obtain a new hyperbolic model, which has advantage of being numerically less costly and providing improved dispersive properties.

Setup and assumption

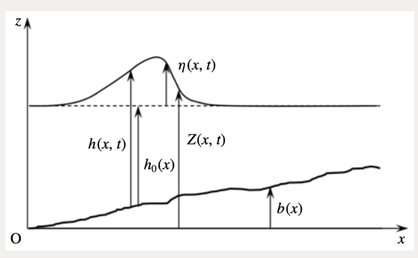


Figure 1. Setup of the model

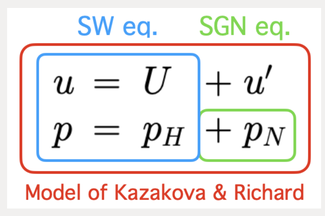


Figure 2. Assumptions in the variables

Figure 1 is a demonstration of two-dimensional flow over a bottom b. The velocity components are denoted as u, w for Ox, Oz direction, respectively.

Assumptions:

- No surface tension and shear stress at the free surface
- The vertical variation of the filtered horizontal velocity is arbitrary (given by u')
- Hypothesis of weakly turbulent flow from Teshukov (2007) gives the order of u' as $O(\varepsilon)$ to include both shearing and dispersion
- The non-hydrostatic pressure is taken into account
- Sound velocity, a, is uniform and constant

Derivation

The model is derived by averaging the filtered incompressible Navier-Stokes equation in the vertical direction and neglecting all the terms of high order with respect to the shallowness parameter

$$\varepsilon = \frac{h_0}{L} \ll 1,$$

where L, h_0 are the horizontal and vertical characteristic length, respectively.

Boundary conditions:

- No-penetration condition at the bottom
- Kinematic and dynamic condition at the free surface

Steps:

1. Averaging the filtered equation over the depth (Mass, Momentum, Acoustic energy, and Mechanical energy equations)
2. As presented in figure 2. The filtered horizontal velocity u is decomposed into its mean velocity U and a deviation part u'. The pressure p is written as the sum of hydrostatic and non-hydrostatic pressure, p_H and p_N.
3. To close the system, new variables are introduced. The enstrophy φ and the depth-averaged non-hydrostatic pressure P are defined as
$$\varphi = \frac{1}{h^3}\int_b^Z (u')^2 dz, \;\; P = \frac{1}{h}\int_b^Z \frac{p_N}{\rho} dz$$
4. Postulate the averaged acoustic energy, dissipation, and the turbulent viscosity with empirical law

Final system of equation in mild slope

In the condition of mild slope, we neglect the terms involving $\partial b/\partial x$ of order less than $O(\varepsilon^3)$.

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0 \\
&\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + hP\right) = \frac{\partial}{\partial x}\left(\frac{2h^3\sqrt{\varphi}}{R}\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x} \\
&\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P - \frac{6hW\sqrt{\varphi}}{R} \\
&\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -\frac{a^2}{r^2}\left(h\frac{\partial U}{\partial x} + 2W\right) \\
&\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -C_r h\varphi^{3/2} + \frac{4h\sqrt{\varphi}}{R}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{16W^2\sqrt{\varphi}}{hR}
\end{aligned} \tag{1}$$

where R is turbulent viscocity parameter, r is a shape factor of the distribution of the non-hydrostatic pressure in the fluid depth, and C_r is a dissipation parameter. Note that the terms highlight in blue are shallow water equation and the terms highlight in green are Hyperbolic SGN model developed by Richard (2021) in incompressible context.

Eigenstructure

In the equation for W and φ, we use the relation $\frac{\partial U}{\partial x} \simeq -\frac{2W}{h}$ to obtain a better eigenstructure.
The eigenvalues of the Jacobian matrix of (1) are real

$$\lambda = U(\text{triple roots}),\; U \pm \sqrt{gh + 3h^2\varphi + P + \frac{a^2}{r^2}}$$

and the corresponding eigenvectors form a basis in $\mathbb{R}^5$, that make the matrix diagonalizable. This guarantees the hyperbolic structure of model (1).

Solitary wave

Solitary wave is a self-similar solution which propagates at a constant phase velocity, c. The model (1) admits a solitary wave when there is no dissipation and the bottom is flat.
Let $\xi = x - ct$ be the phase. The boundary conditions at infinity are

$$h(x=\infty) = h_\infty,\; U(x=\infty) = 0,\; W(x=\infty) = 0,$$
$$P(x=\infty) = 0,\; \varphi(x=\infty) = \varphi_\infty.$$

The solitary wave of speed c for (1) is a mapping

$$(x,t) \in \mathbb{R}^2 \mapsto (h,U,W,P,\varphi)(\xi) = (h,U,W,P,\varphi)(x-ct),$$

satisfy

$$U = c\left(1 - \frac{h_\infty}{h}\right),\; P = \frac{C_2}{h} - h^2\varphi_\infty - \frac{gh}{2} - \frac{C_1^2}{h^2},\; \varphi = \varphi_\infty$$

where $C_1 = -ch_\infty$ and $C_2 = \frac{C_1^2}{h_\infty} + \frac{gh_\infty^2}{2} + h_\infty^3\varphi_\infty$ and (h, W) solves the ODE

$$\begin{cases} h' = \dfrac{2W}{C_1}\left(\dfrac{1}{h} - \dfrac{r^2}{a^2}\dfrac{dP}{dh}\right)^{-1} \\ W' = -\dfrac{3}{2C_1}P \end{cases} \tag{2}$$

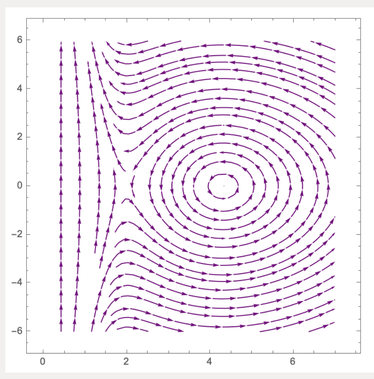


Figure 3. phase portrait of (2)

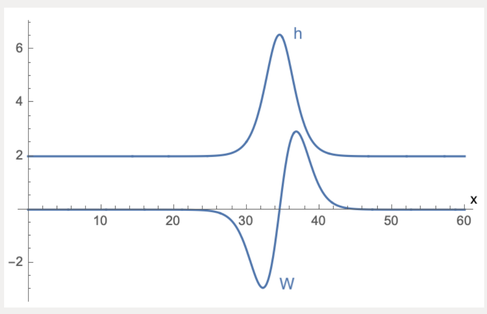


Figure 4. Profile of h and W

Perspectives

- The derivation will be generalised to 2D.
- New model will be coupled with sediment transport equations using two important features.
  - The model is hyperbolic, which is simpler for numerical resolution.
  - The enstrophy (vertical variations of horizontal velocity) is related to vorticity and turbulence due to wave breaking.

References

Richard, G. L. (2021). "An extension of the Boussinesq-type models to weakly compressible flows". In: *European Journal of Mechanics - B/Fluids* 89, pp. 217–240. ISSN: 0997-7546. DOI: 10.1016/j.euromechflu.2021.05.011.

Kazakova, M. and G. L. Richard (2019). "A new model of shoaling and breaking waves: one-dimensional solitary wave on a mild sloping beach". In: *Journal of Fluid Mechanics* 862, pp. 552–591. DOI: 10.1017/jfm.2018.947.