Hyperbolic dispersive breaking wave model with improved breaking criterion

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Introduction

Wave breaking is known to be a complex phenomenon where the internal mechanism hasn't yet been fully studied. As fully resolving the interior of the fluid can be time-consuming, our goal is to derive a simplified model.

Objectives:

- Describe wave breaking phenomenon (Kazakova and Richard 2019)
- Preserve and improve the dispersive properties
- Have a hyperbolic reformulation (Richard 2021) to avoid dealing with mixed space-time high order derivative
- Have a robust and predictive breaking criterion to indicate breaking zone.

Setup and assumption

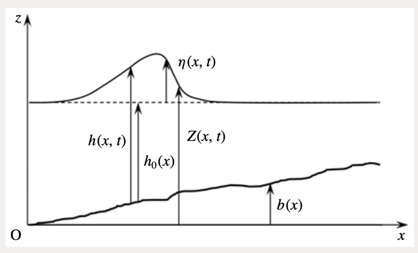


Figure 1. Setup of the model

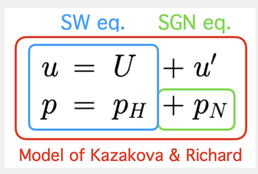


Figure 2. Assumptions in the variables

Assumptions:

- No surface tension and shear stress at the free surface
- The vertical variation of the filtered horizontal velocity is arbitrary (given by u')
- Hypothesis of weakly turbulent flow from Teshukov (2007) gives the order of u' as $O(\varepsilon)$ to include both shearing and dispersion
- The non-hydrostatic pressure is taken into account
- Sound velocity, a_c, is uniform and constant

Derivation

Asymptotic analysis with respect to the shallowness parameter

$$\varepsilon = \frac{h_0}{L} \ll 1,$$

where L, h_0 are the horizontal and vertical characteristic length, respectively.

Boundary conditions:

- No-penetration condition at the bottom
- Kinematic and dynamic condition at the free surface

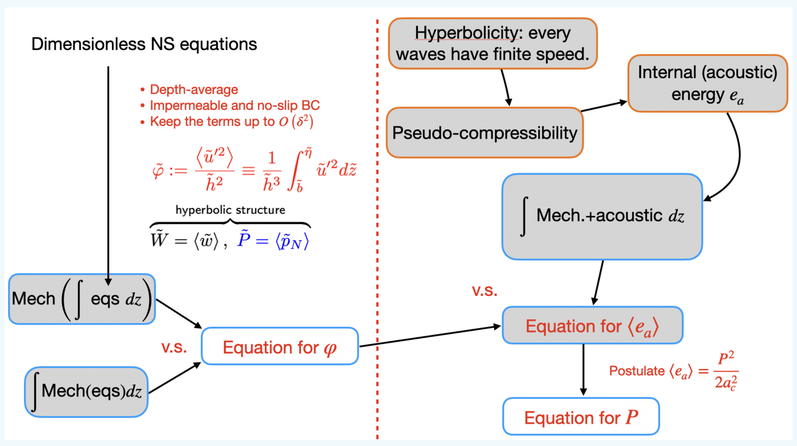


Figure 3. Sketch of the derivation

Final system of equation in mild slope

Under the mild slope assumption, we neglect the terms involving $\partial b/\partial x$ of order greater than $O(\varepsilon^3)$.

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0 \\
&\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + hP\right) = \frac{\partial}{\partial x}\left(2\nu_T h\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x} \\
&\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P + 3\nu_T\frac{\partial U}{\partial x} \\
&\frac{\partial hP}{\partial t} + \frac{\partial hUP}{\partial x} = -a_c^2\left(h\frac{\partial U}{\partial x} + 2W\right) \\
&\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -\frac{2}{h}\langle P^r\rangle + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 - \frac{8\nu_T W}{h^2}\frac{\partial U}{\partial x}
\end{aligned} \tag{1}$$

Blue: Shallow Water equation

Green: Incompressible hyperbolic SGN model developed by Richard (2021)

- The conservative part of the model is hyperbolic if $h > 0$
- Asymptotic dispersive properties to the SGN equation when $a_c \to \infty$
- A predictive law associated with breaking criterion is developed by Hung 2025
- 1D model with improved dispersive properties and 2D model have been derived

Virtual enstrophy ψ and breaking criterion

If the criterion is based on only one enstrophy $\longrightarrow$ strong discontinuity in the flux $\longrightarrow$ introduce the virtual enstrophy ψ

$$\begin{aligned}
&\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left(hU^2 + \frac{gh^2}{2} + h^3\varphi + hP\right) = \frac{\partial}{\partial x}\left(2\nu_T h\frac{\partial U}{\partial x}\right) - gh\frac{\partial b}{\partial x} \\
&\frac{\partial hW}{\partial t} + \frac{\partial hUW}{\partial x} = \frac{3}{2}P - \frac{6W\nu_T}{h} \\
&\frac{\partial h\varphi}{\partial t} + \frac{\partial hU\varphi}{\partial x} = -C_r h\varphi^{3/2} + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{16W^2\nu_T}{h^3} \\
&\frac{\partial h\psi}{\partial t} + \frac{\partial hU\psi}{\partial x} = -C_r h\psi^{3/2} + \frac{4\nu_T}{h}\left(\frac{\partial U}{\partial x}\right)^2 + \frac{16W^2\nu_T}{h^3}
\end{aligned}$$

Energy dissipation only activates when the criterion meets.

- Track the vorticity generated inside the fluid
- Does not affect the original system when there's no breaking

Previous breaking criterion (Kazakova and Richard 2019):

$$\psi = \frac{g}{h_0}\tilde{\psi}, \quad \tilde{\psi}_0 = \begin{cases} 0.1 + \dfrac{0.031}{\mu_0}, & \mu_0 > 0.05 \\ 0, & \mu_0 < 0.05 \end{cases}$$

Activate if $\tilde{\psi} > \tilde{\psi}_0$

(μ_0 : nonlinearity of the initial wave)

Improved breaking criterion (Hung 2025): local slope-dependent

$$\tilde{\psi}_\eta = \frac{\psi\eta}{g} < \tilde{\psi}_h = \frac{\psi h}{g},$$

with $\tilde{\psi}_0 > \tilde{\psi}_1 \geq 0,\ \tilde{\psi}_i = \tilde{\psi}_i(\tan\beta)$

Activate if $\tilde{\psi}_\eta > \tilde{\psi}_0$ and deactivate if $\tilde{\psi}_h < \tilde{\psi}_1$

Visualize the criterion

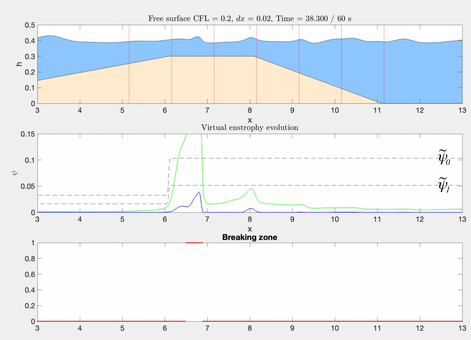


Figure 4. Visualization of the improved breaking criterion

- Activation: $\tilde{\psi}_\eta$ reaches $\tilde{\psi}_0$
- The breaking will terminate only after $\tilde{\psi}_h$ becomes smaller than $\tilde{\psi}_1$
- The two thresholds method avoids an abrupt deactivation after breaking
- The slope-dependent law for thresholds are obtained empirically with solitary wave test cases but are found to be applicable in more complex wave train cases.

Hydrodynamic validation

The hydrodynamic model has been validated through many numerical test cases with various scenarios.

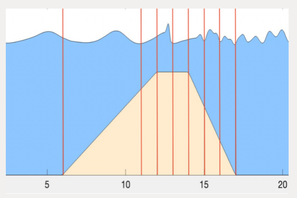


Figure 5. Experiment of Beji & Battjes consists of wave trains propagate over a submerged bar

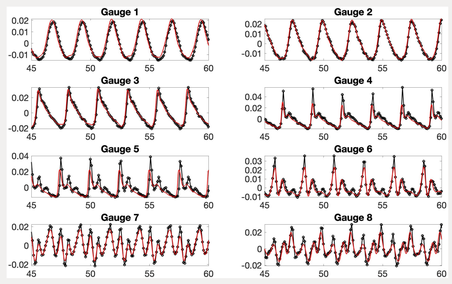


Figure 6. Time series comparison of the amplpitude at the gauge points. Numerical results in red solid line and experimental data in black markers.

Perspectives

- The coupling of the sediment transport equation with the hydrodynamic model is being studied at the moment.
- Implementation of hydrodynamic model within TOLOSA using a new numerical scheme developed by Richard & Duran, 2024 for operational applications.

References

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- Richard, G. L. (2021). "An extension of the Boussinesq-type models to weakly compressible flows". In: *European Journal of Mechanics - B/Fluids* 89, pp. 217–240. ISSN: 0997-7546. DOI: 10.1016/j.euromechflu.2021.05.011.
- Kazakova, M. and G. L. Richard (2019). "A new model of shoaling and breaking waves: one-dimensional solitary wave on a mild sloping beach". In: *Journal of Fluid Mechanics* 862, pp. 552–591. DOI: 10.1017/jfm.2018.947.